arXiv:physics/0610281 v1 31 Oct 2006

Vertex labeling and routing in expanded Apollonian networks

Z. Zhang, F. Comellas, G. Fertin, A. Raspaud, L. Rong, S. Zhou

Zhongzhi Zhang

Department of Computer Science and Engineering and

Shanghai Key Lab of Intelligent Information Processing,

Fudan University, Shanghai 200433, China

zhangzz@fudan.edu.cn

Francesc Comellas

Dep. de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya

Av. Canal Olímpic s/n, 08860 Castelldefels, Barcelona, Catalonia, Spain

comellas@ma4.upc.edu

Guillaume Fertin

LINA, Université de Nantes

2 rue de la Houssinière, BP 92208, 44322 Nantes Cedex 3, France

fertin@lina.univ-nantes.fr

André Raspaud

LaBRI, Université Bordeaux 1

351, cours de la Libération, 33405 Talence Cedex, France

raspaud@labri.fr

Lili Rong

Institute of Systems Engineering, Dalian University of Technology

Dalian 116024, Liaoning, China

llrong@dlut.edu.cn

Shuigeng Zhou

Department of Computer Science and Engineering and

Shanghai Key Lab of Intelligent Information Processing,

Fudan University, Shanghai 200433, China

sgzhou@fudan.edu.cn

Abstract

We present a family of networks, expanded deterministic Apollonian networks, which are a generalization of the Apollonian networks and are simultaneously scale-free, small-world, and highly clustered. We introduce a labeling of their vertices that allows to determine a shortest path routing between any two vertices of the network based only on the labels.

1 Introduction

In these last few years there has been a growing interest in the study of complex networks [2, 20, 31, 11], which can help to describe many social, biological, and communication systems, such as co-author networks [30], sexual networks [29], metabolic networks [27], protein networks in the cell [26], Internet [23], and the World Wide Web [4]. Extensive observational studies show that many real-life networks have at least three important common statistical characteristics: the degree distribution exhibits a power law tail with an exponent taking a value between 2 and 3 (*scale-free*); nodes having a common neighbor are far more likely to be linked to each other than are two nodes selected randomly (*highly clustered*); the expected number of links needed to go from one arbitrarily selected node to another one is low (*small-world property*).

These empirical findings have lead to a new kind of network models [2, 20, 31, 11]. The research on these new models was started by the two seminal papers by Watts and Strogatz on small-world networks [32] and Barabási and Albert on scale-free networks [7]. A wide variety of network models and mechanisms, including initial attractiveness [21], nonlinear preferential attachment [28], aging and cost [5], competitive dynamics [9], edge rewiring [3] and removal [19], duplication [12], which may represent processes realistically taking place in real-life systems, have been proposed.

Recently, based on the classical Apollonian packing, Andrade *et al.* introduced Apollonian networks [6] which were simultaneously proposed by Doye and Massen in [22]. Apollonian networks belong to a deterministic type of networks studied earlier in Refs. [8, 17, 18, 16, 15] which have received much interest recently [40, 35, 33, 37, 38]. Two-dimensional Apollonian networks are simultaneously scale-free, small-world, Euclidean, space filling, and with matching graphs [6, 40]. They may provide valuable insight into real-life networks; moreover, they are maximal planar graphs and this property is of particular interest for the layout of printed circuits and related problems [6, 40]. More recently, some interesting dynamical processes [40, 37, 25], such as percolation [40], epidemic spreading [40], synchronization [37], and random walks [25], taking place on these networks have been also investigated.

Networks are composed of vertices (nodes) and edges (links) and are very often studied considering branch of discrete mathematics known as graph theory. One active subject in graph theory is graph labeling [24]. This is not only due to its theoretical importance but also because of the wide range of applications in many fields [10], such as x-rays, crystallography, coding theory, radar, astronomy, circuit design, and communication design.

In this paper we present an extension of the general high dimensional Apollonian networks [6, 22, 33] which includes the deterministic small-world network introduced in [36]. We give a vertex labeling, so that queries for the shortest path between any two vertices can be efficiently answered thanks to it. Finding shortest paths in networks is a well-studied and important problem with also many applications [1]. Our labeling may be useful in aspects such as network optimization and information dissemination, which are directly related to the problem of finding shortest paths between all pairs of vertices of the network.

2 Expanded Apollonian networks

In this section we present a network model defined in a iterative way. The model, which we call *expanded Apollonian network* (EAN), is an extension of the general high dimensional Apollonian network [6, 22, 33] which includes the deterministic small-world network introduced in [36].

The networks, denoted by $A(d,t)$ after t iterations with $d \geq 1$ and $t \geq 0$, are constructed as follows. For $t = 0$, $A(d,0)$ is a complete graph K_{d+2} (or $(d+2)$-clique). For $t \geq 1$, $A(d,t)$ is obtained from $A(d,t-1)$. For each of the existing subgraphs of $A(d,t-1)$ that is isomorphic to a $(d+1)$-clique and created at step $t-1$, a new vertex is created and connected to all the vertices of this subgraph. Figure 1 shows the network growing process for the particular case where $d = 2$.

Let $n_v(t)$ and $n_e(t)$ denote the number of vertices and edges created at step t, respectively. According to the network construction, one can see that at step t_i ($t_i > 1$) the number of newly introduced vertices and edges

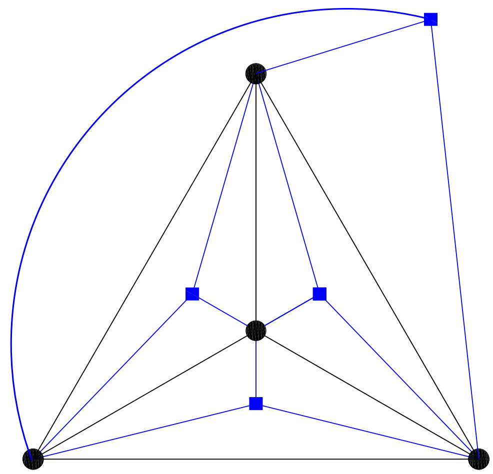

Figure 1: Illustration of a growing network in the case of $d = 2$, showing the first two steps of growing process.

is $n_v(t_i) = (d+2)(d+1)^{t_i-1}$ and $n_e(t_i) = (d+2)(d+1)^{t_i}$. From these results, we can easily compute the total number of vertices N_t and edges E_t at step t, which are $N_t = \frac{(d+2)[(d+1)^t+d-1]}{d}$ and $E_t = (d+2)(d+1)\frac{2(d+1)^t+d-2}{2d}$, respectively. So for large t, the average degree $\overline{k}_t = \frac{2E_t}{N_t}$ is approximately $2(d+1)$.

This general model includes existing models, as listed below.

Indeed, when $d = 1$, the network is the deterministic small-world network (DSWN) introduced in [36] and further generalized in [34]. DSWN is an exponential network, its degree distribution $P(k)$ is an exponential of a power of degree k. For a node of degree k, the exact clustering coefficient is $\frac{2}{k}$. The average clustering coefficient of DSWN is $\ln 2$, which approaches to a high constant value 0.6931. The average path length of DSWN grows logarithmically with the number of network vertices [39].

When $d \geq 2$, the networks are exactly the same as the high-dimensional Apollonian networks (HDAN) with d indicating the dimension [6, 22, 33, 38]. HDAN present the typical characteristics of real-life networks in nature and society, and their main topological properties are controlled by dimension d. They have a power law degree distribution with exponent $\gamma = 1 + \frac{\ln(d+1)}{\ln d}$ belonging to the interval between 2 and 3 [6, 22, 33, 38]. For any individual vertex in HDAN, its clustering coefficient $C(k)$ is also inversely proportional to its degree k as $C(k) = \frac{2d(k-\frac{d+1}{2})}{k(k-1)}$. The mean value C of clustering coefficient of all vertices in HDAN is very large and is an increasing function of d. For instance, in the special cases where $d = 2$ and $d = 3$, C asymptotically reaches values 0.8284 and 0.8852, respectively. In addition, HDAN are small worlds. The diameter of HDAN, defined as the longest shortest distance between all pairs of vertices, increases logarithmically with the number of vertices. So, the EAN model exhibits a transition from an exponential network ($d = 1$) to scale-free networks ($d \geq 2$).

3 Vertex labeling

Vertex labeling of a network is an assignment of labels to all the vertices in the network. In most applications, labels are nonnegative integers, though in general real numbers could be used [24]. In this section, we describe a way to label the vertices of $A(d,t)$, for any $d \geq 1$ and $t \geq 0$, such that a routing by shortest paths between any two vertices of $A(d,t)$ can be deduced from the labels. We note that a more general result on shortest paths routing of graphs with given treewidth is given in [13]. However, here we address the more specific case of the expanded Apollonian networks $A(d,t)$. In what follows, we will denote $L(v)$ as the label of vertex v, for any vertex v belonging to $A(d,t)$.

Here the labeling idea, inspired from [14], is to assign to any vertex v created at step $t \geq 1$ a label of length t, in the form of a word of t digits, each digit being an integer between 1 and $d+2$ (the vertices obtained at step $t = 0$, i.e. the vertices of the *initial* $(d+2)$-clique $A(d,0)$, are assigned a special label). More precisely, the labeling of any vertex v of $A(d,t)$ is done thanks to the following rules:

- Label the vertices of the initial $(d+2)$-clique $A(d,0)$ arbitrarily, with labels $1', 2' \ldots (d+2)'$.

- At any step $t \geq 1$, when a new vertex v is added and joined to all vertices of a clique K_{d+1}:

  1. If v is connected to $d+1$ vertices of the initial $(d+2)$-clique, then $L(v) = l$, where l' is the only vertex of the initial $(d+2)$-clique that does not belong to this $(d+1)$-clique.

2. If not, then v is connected to $w_1, w_2 \dots w_{d+1}$, where at least one of the w_i's is not a vertex of the initial $(d+2)$-clique. Thus, any such vertex has a label $L(w_i) = s_{1,i}s_{2,i}\dots s_{k,i}$. W.l.o.g., let w_1 be the vertex not belonging to the initial $(d+2)$-clique with the longest label. In that case, we give vertex v the label $L(v)$ defined as follows: $L(v) = \alpha \cdot L(w_1)$, where $1 \leq \alpha \leq d+2$ is the only integer not appearing as first digit in the labels of $w_1, w_2 \dots w_{d+1}$, that is $\alpha = \{1, 2 \dots d, d+1, d+2\} / \cup_{i=1}^{d+1} s_{1,i}$ (the fact that α is unique will be proved by Property 1 below).

Such a labeling is illustrated in Figure 2. In the upper part of this figure, we label the vertices of $A(d,t)$, for $d = 1$ and up to $t = 3$. We see that vertex u, created at step 1, has label $L(u) = 2$ because it is not connected to vertex $2'$ of the initial 3-clique (triangle). Vertex w is is not connected to any vertex of the initial 3-clique, its label is first composed of the only digit not appearing as first digit of its neighbors (in this case, 1), concatenated with the longest label of its neighbors (in this case, 23). Analogously, in the lower part of Figure 2 , where, for sake of clarity, only a part of $A(2,3)$ is drawn, the vertices have been labeled. For the same reasons, we can see that vertex u has label $L(u) = 4$, while w has label $L(w) = 234$.

Thus, we see that for any $t \geq 1$, any vertex v_t created at step t has a unique label, and that for any vertex v created at step $t \geq 1$, $L(v) = s_1 s_2 \dots s_t$ is of length t, where each digit s_j satisfies $1 \leq s_j \leq d+2$; while the vertices created at step 0 have length 1 (these are the l', $1 \leq l \leq d+2$).

We note that since for any step $t \geq 1$, the number of vertices that are added to the expanded Apollonian networks is equal to $(d+2)(d+1)^{t-1}$, the labeling we propose is optimal in the sense that each label $L(v_t)$ of a vertex created at step t is a $(d+2)$-ary word of length t. Globally, any vertex of $A(d,t)$ is assigned a label of length $O(\log_{d+2} t)$; since there are $N_t = (d+2)\frac{(d+1)^t+d-1}{d}$ vertices in $A(d,t)$, we can see that, overall, the labeling is optimal as well.

Next, we give three properties about the above labeling. Property 1 ensures that our labeling is deterministic. Property 2 is a tool to prove Property 3, the latter being important to show that our routing protocol is valid and of shortest paths.

Property 1 In $A(d,t)$, for any $(d+2)$-clique induced by vertices w_1, w_2 $\dots$ w_{d+2}, every integer $1 \leq i \leq d+2$ appears exactly once as the first digit of the label of a w_j.

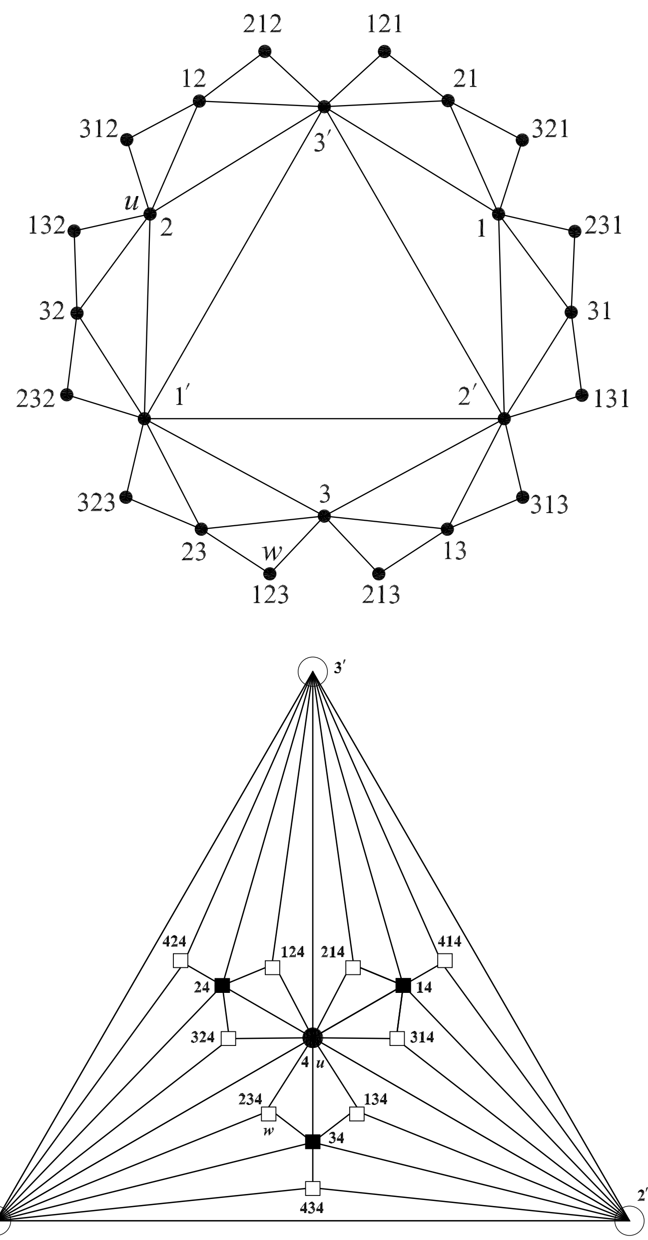

Figure 2: (Above) Labels of all vertices of $A(1,3)$. (Below) Labels of a part of the vertices of $A(2,3)$

.

Proof. By induction on t. When $t = 1$, the property is true by construction. Suppose now that the property is true for any $t' < t$, and let us then show it is true for t. Any $(d+2)$-clique in $A(d,t)$ is composed of exactly one vertex v created at a given step t_1, and $d+1$ vertices $w_1, w_2, \ldots w_{d+1}$ created at steps strictly less than t_1. If $t_1 < t$, then the property is true by induction hypothesis. If $t_1 = t$, we suppose that w_{d+2} is connected to a $(d+1)$-clique $\mathcal{C}$ composed of $w_1, w_2, \ldots w_{d+1}$. It is clear that $\mathcal{C}$ did not exist at step $t-1$. In other words, one of the w_i's, say w_1, has been created at step $t-1$, based on $d+1$ vertices $w_2, w_3 \ldots w_{d+1}$ and x. By induction hypothesis, each integer $1 \leq i \leq d+2$ appears exactly once as first digit of the labels of $w_1, w_2, w_3 \ldots w_{d+1}, x$. However, by construction, the first digit of $L(w_{d+2})$ is the first digit of $L(x)$. Thus we conclude that each integer $1 \leq i \leq d+2$ also appears exactly once as first digit of the labels of $w_1, w_2, w_3 \ldots w_{d+1}, w_{d+2}$, and the result is proved by induction.

Property 2 Let v_t be a vertex of $A(d,t)$ created at step $t \geq 1$. Among the vertices $w_1, w_2, \ldots w_{d+1}$ forming the $(d+1)$-clique that generated v_t, let $w_1, w_2 \ldots w_k$, $k \leq d+1$, be the vertices that do not belong to the initial $(d+2)$-clique. Then $L(v_t)$ is a superstring of $L(w_i)$ for all $1 \leq i \leq k$.

Proof. By induction on t. When $t = 1$, any vertex v_1 created at step 1 is connected to vertices of the initial $(d+2)$-clique only. Thus the result is true. Now suppose the result is true for any $1 \leq t' \leq t-1$, $t \geq 2$, and let us prove it is then true for t. For this, we consider a vertex v_t created at step t, and the $(d+1)$-clique $\mathcal{C}$ it is connected to. Suppose v_t is a neighbor of w_p which was created at step $t-1$. However, w_p was created itself thanks to a $(d+1)$-clique, say $\mathcal{C}'$, composed of vertices $x_1, x_2 \ldots x_{d+1}$. W.l.o.g., suppose that $k \leq d+1$ such vertices, $x_1, x_2 \ldots x_k$ do not belong to the initial $(d+2)$-clique. By induction hypothesis, $L(x_i) \subseteq L(w_p)$ for any $1 \leq i \leq k$. Hence, in $\mathcal{C}$, w_p is the vertex not belonging to the initial $(d+2)$-clique that has the longest label. By construction of $L(v_t)$, we have that $L(w_p) \subseteq L(v_t)$, thus we also conclude that $L(x_i) \subseteq L(v_t)$ for any $1 \leq i \leq k$. Thus $L(v_t)$ is a superstring of the labels of any vertex of $\mathcal{C}$ that does not belong to the initial $(d+2)$-clique, and the result is proved by induction.

Property 3 Let v_t be a vertex of $A(d,t)$ created at step $t \geq 1$. For any $1 \leq i \leq d+2$, if $i \notin L(v_t)$, then v_t is a neighbor of a vertex v' of the initial $(d+2)$-clique, such that $L(v') = i'$.

Proof. By induction on t. When $t = 1$, any vertex v_1 constructed at step 1 is assigned label i, where i' is the only vertex of the initial $(d+2)$-clique v_t is not connected to ; thus, by construction, the property is satisfied.

Now we suppose that the property is true for any $1 \leq t' \leq t-1$, $t \geq 2$, and we will show it then holds for t as well. As for the previous property, we consider a vertex v_t created at step t, and the $(d+1)$-clique $\mathcal{C}$ it is connected to.

Suppose v_t is connected to a vertex w_{t-1} that was created at step $t-1$. However, w_{t-1} was created itself thanks to a $(d+1)$-clique $\mathcal{C}'$ composed of vertices $x_1, x_2 \dots x_{d+1}$. Among those $d+1$ vertices, only one, say x_p, does not belong to $\mathcal{C}$. W.l.o.g., suppose that $k \leq d+1$ such vertices, $x_1, x_2 \dots x_k$ do not belong to the initial $(d+2)$-clique. Now suppose that $i \notin L(v_t)$; then i appears as the first digit of one of the $L(x_j)$'s, $j \in [1, p-1] \bigcup [p+1, d+1]$, or of $L(w_{t-1})$ (by Property 1). However, $L(x_j) \subseteq L(w_{t-1}) \subseteq L(v_t)$ for any $1 \leq j \leq k$ (by Property 2). Thus, neither w_{t-1} nor any vertex among the x_j's, $1 \leq j \leq k$ contains the digit i in its label. Hence, only a vertex y from the initial $(d+2)$-clique can have i in its label, and thus $L(y) = i'$. Hence it suffices to show that v_t and y are neighbors to prove the property. The only case for which this would not happen is when $y = x_p$; we will show that this is not possible. Indeed, by construction of the labels, the first digit of $L(v_t)$ is the only integer not appearing as first digit of the labels of the vertices of $\mathcal{C}$, that is $w_{t-1}, x_1, x_2 \dots x_{p-1}, x_{p+1} \dots x_{d+1}$. However, the fact that we suppose $y = x_p$ means that no vertex of $\mathcal{C}$ contains i in its label. Thus this would mean that the first digit of $L(v_t)$ is i, a contradiction. Thus, v_t is connected to y with $L(y) = i'$, and the induction is proved.

4 Routing by shortest path

Now we describe the routing protocol between any two vertices u and v of $A(d,t)$, with labels respectively equal to $L(u)$ and $L(v)$. We note that since $A(d,0)$ is isomorphic to the complete graph K_{d+2}, we can assume $t \geq 1$. The routing protocol is special here in the sense that the routing is done both from u and v, until they reach a common vertex. Hence, the routing strategy will be used simultaneously from u and from v. In order to find a shortest path between any two vertices u and v, the routing protocol is as follows. First we compute the longest common suffix $LCS(L(u), L(v))$ of $L(u)$ and $L(v)$, then we distinguish two cases:

1. If $LCS(L(u), L(v)) = \emptyset$:

    (a) Simultaneously from u and v (say, from u): let $u = u_0$ and go from

u_i to u_{i+1}, $i \geq 0$ where u_{i+1} is the neighbor of u_i with shortest label.

(b) Stop when u_k is a neighbor of the initial $(d+2)$-clique.
Let $\bar{L}(u_k)$ (resp. $\bar{L}(v_{k'})$) be the integers not present in $L(u_k)$ (resp. $L(v_{k'})$), and let $S = \bar{L}(u_k) \cap \bar{L}(v_{k'})$.

i. If $S \neq \emptyset$, pick any $l \in S$, and close the path by taking the edge from u_k to l', and the edge from l' to $v_{k'}$.

ii. If $S = \emptyset$, route from u_k to any neighbor l'_1 (belonging to the initial $(d+2)$-clique) of u_k, and do similarly from $v_{k'}$ to a neighbor l'_2 (belonging to the initial $(d+2)$-clique) of $v_{k'}$. Then, take the edge from l'_1 to l'_2 and thus close the path from u to v.

2. If $LCS(L(u), L(v)) \neq \emptyset$, then let us call *least common clique* of u and v, or $LCC(u, v)$, the $(d+2)$-clique composed of the vertex with label $LCS(L(u), L(v))$ and the $d+1$ vertices forming the $(d+1)$-clique that generated the vertex of label $LCS(L(u), L(v))$. We simultaneously route from u and v to (respectively) u_k and $v_{k'}$, going each time to the neighbor with $LCS(L(u), L(v))$ as label suffix, and having the shortest label. Similarly as above, we stop at u_k (resp. $v_{k'}$), where u_k (resp. $v_{k'}$) is the first of the u_i's (resp. of the v_j's) to be a neighbor of $LCC(u, v)$. Then there are two subcases, depending on $Q = L(u_k) \cap L(v_{k'})$.

(a) If $Q \neq \emptyset$, close the path by going to any vertex w with label l, $l \in Q$.

(b) If $Q = \emptyset$, then route from u_k (resp. $v_{k'}$) to any neighbor w_1 (resp. w_2) in $LCC(u, v)$, and close the path by taking the edge (w_1, w_2), which exists since both vertices w_1 and w_2 belong to the same clique $LCC(u, v)$.

Proposition 1 The above mentioned routing algorithm is valid, and of shortest paths.

Proof. Let us first give the main ideas for the validity of the above routing protocol. Take any two vertices u and v. By construction of $L(u)$ and $L(v)$, the longest common suffix $LCS(L(u), L(v))$ indicates to which $(d+2)$-clique u and v have to go. We can consider this as a way for u and v to reach their least common ancestor in the graph of cliques induced by the construction of $A(d,t)$, or the "*least common clique*". In Case (i), this least common clique is the initial $(d+2)$-clique ; thus, u and v have to get back to it. In Case (ii), the shortest path does not go through the initial $(d+2)$-clique, and the least common clique of u and v, say $LCC(u,v)$, is indicated by the longest common suffix $LCS(L(u),L(v))$. In other words, the length of $LCS(L(u),L(v))$ indicates the depth of $LCC(u,v)$ in the graph of cliques induced by the construction of $A(d,t)$. In that case, the routing is similar as in Case (i), except that the initial $(d+2)$-clique has to be replaced by the clique $LCC(u,v)$. Hence, the idea is to adopt the same kind of routing, considering only neighbors which also have $LCS(L(u),L(v))$ as suffix in their labels.

When this least common ancestor is determined, one can see, still by construction, that the shortest route to reach this clique (either from u or v) is to go to the neighbor which has smallest label, since the length of the label indicates at which step the vertex was created. Indeed, the earlier the neighbor w was created, the smaller the distance from w to the least common clique is.

After we have reached, from u (resp. from v), a vertex u_k (resp. $v_{k'}$) that is a neighbor of the least common clique, the last thing we need to know is whether u_k and $v_{k'}$ are neighbors. Thanks to Property 3, we know that looking at $L(u_k)$ and $L(v_{k'})$ is sufficient to answer this question. More precisely:

- In Case (i)(b)-1, u_k and $v_{k'}$ share a neighbor in the initial $(d+2)$-clique (by Property 3). All those common neighbors have label l', where $l \in S$. Hence, if we pick any $l \in S$, then there exists and edge between u_k and l', as well as an edge between l' and $v_{k'}$.

- In Case (i)(b)-2, u_k and $v_{k'}$ do not share a neighbor in the initial $(d+2)$-clique. Hence, taking a route from u_k (resp. $v_{k'}$) to any neighbor l'_1 (resp. l'_2) belonging to the initial $(d+2)$-clique, we can finally take the edge from l'_1 to l'_2 (which are neighbors, since they both belong to the initial $(d+2)$-clique) in order to close the path from u to v.

- In Case (ii)(a), u_k and $v_{k'}$ share a neighbor in $LCC(u,v)$. Hence we can close the path by going to any vertex w with label l, $l \in Q$, since w is a neighbor of both u_k and $v_{k'}$.

- In Case (ii)(b), u_k and $v_{k'}$ do not share a neighbor in $LCC(u,v)$. Hence we route from u_k (resp. $v_{k'}$) to any neighbor w_1 (resp. w_2) in $LCC(u,v)$, and we close the path by taking the edge (w_1, w_2). This edge exists since both vertices w_1 and w_2 belong to the same clique $LCC(u, v)$.

Hence we conclude that our labeling of vertices in $A(d, t)$ allows a routing between any two vertices u and v, and that it is of shortest paths.

5 Conclusion

We have proposed an expanded deterministic Apollonian network model, which represents a transition for degree distribution between exponential and power law distributions. Our model successfully reproduces some remarkable characteristics in many nature and man-made networks. We have also introduced a vertex labeling for these networks. The length of the label is optimal. Using the vertex labels it is possible to find in an efficient way a shortest path between any pair of vertices. Nowadays, efficient handling and delivery in communication networks (e.g. the Internet) has become one important practical issues, and it is directly related to the problem of finding shortest paths between any two vertices. Our results, therefore, can be useful when describing new communication protocols for complex communication systems.

Acknowledgment

Zz.Z. and Sg.Z. gratefully acknowledge partial support from the National Natural Science Foundation of China under Grant Nos. 60373019, 60573183, and 90612007. Support for F.C. was provided by the Secretaria de Estado de Universidades e Investigación (Ministerio de Educación y Ciencia), Spain, and the European Regional Development Fund (ERDF) under project TEC2005-03575.

References

[1] R. Ahuja, T. Magnanti, and J. Orlin, Network Flows, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[2] R. Albert and A.-L. Barabási, Statistical mechanics of complex networks, Rev. Mod. Phys. **74** (2002) 47-97.

[3] R.Albert and A.-L. Barabási, Topology of evolving networks: Local events and universality, Phys. Rev. Lett. **85** (2000) 5234-5237.

[4] R. Albert, H. Jeong, and A.-L. Barabási, Diameter of the world wide web, Nature (London) **401** (1999) 130-131.

[5] L. A. N., Amaral, A. Scala, M. Barthélémy, H. E. Stanley, Classes of small-world networks, Proc. Natl. Acad. Sci. U.S.A. **9** (2000) 11149-11152.

[6] J. S. Andrade Jr., H. J. Herrmann, R. F. S. Andrade and L. R. da Silva, Apollonian Networks: Simultaneously scale-free, small world, Euclidean, space filling, and with matching graphs. Phys. Rev. Lett. **94** (2005) 018702.

[7] A.-L. Barabási and R. Albert, Emergence of scaling in random networks, Science **286** (1999) 509-512.

[8] A.-L. Barabási, E. Ravasz, and T. Vicsek, Deterministic scale-free networks, Physica A **299** (2001) 559-564.

[9] G. Bianconi and A.-L. Barabási, Competition and multiscaling in evolving networks, Europhys. Lett. **54** (2001) 436-442.

[10] G. S. Bloom and S. W. Golomb, Applications of Numbered Undirected Graphs, Proc. IEEE, **5** (1977) 562-570.

[11] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwanga, Complex networks: Structure and dynamics, Phys. Rep. **424** (2006) 175-308.

[12] F. Chung, Linyuan Lu, T. G. Dewey, D. J. Galas, Duplication models for biological networks, J. of Comput. Biology **10** (2003) 677-688.

[13] S. Chaudhuri, C. D. Zaroliagis, Shortest paths in digraphs of small treewidth. Part I: Sequential algorithms. Algorithmica **27** (2000) 212-226.

[14] F. Comellas, G. Fertin, and A. Raspaud, Vertex labeling and routing in recursive clique-trees, a new family of small-world scale-free graphs. The 10th Int. Colloquium on Structural Information and Communication

Complexity. Sirocco 2003. Proceedings in Informatics vol. 17, pp. 73-87. Eds. P. Fraigniaud. J. Sibeyn. Carleton Scientific, 2003 .

[15] F. Comellas, G. Fertin, and A. Raspaud, Recursive graphs with small-world scale-free properties, Phys. Rev. E **69** (2004) 037104.

[16] F. Comellas and M. Sampels, Deterministic small-world networks, Physica A **309** (2002) 231-235.

[17] F. Comellas, J. Ozón, and J.G. Peters, Deterministic small-world communication networks, Inf. Process. Lett. **76** (2000) 83-90.

[18] S.N. Dorogovtsev, A.V. Goltsev, and J.F.F. Mendes, Pseudofractal scale-free web, Phys. Rev. E **65** (2002) 066122.

[19] S.N. Dorogovtsev and J.F.F. Mendes, Scaling behaviour of developing and decaying networks, Europhys. Lett. **52** (2000) 33-39.

[20] S.N. Dorogvtsev and J.F.F. Mendes, Evolution of networks, Adv. Phys. **51** (2002) 1079-1187.

[21] S.N. Dorogovtsev, J.F.F. Mendes, A.N. Samukhin, Structure of growing networks with preferential linking, Phys. Rev. Lett. **85** (2000) 4633-4636.

[22] J. P. K. Doye and C. P. Massen. Self-similar disk packings as model spatial scale-free networks. Phys. Rev. E **71** (2005) 01628.

[23] M. Faloutsos, P. Faloutsos, and C. Faloutsos, On power-law relationships of the internet topology, Comput. Commun. Rev. **29** (1999) 251-260.

[24] J. A. Gallian, A dynamic survey of graph labeling, The electronic journal of combinatorics, **5**, (2005) #DS6.

[25] Z.-G. Huang, X.-J. Xu, Z.-X. Wu, and Y.-H. Wang, Walks on Apollonian networks, Eur. Phys. J B **51** (2006) 549-553.

[26] H. Jeong, S. Mason, A.-L. Barabási, and Z.N. Oltvai, Lethality and centrality in protein networks, Nature **411** (2001) 41-42.

[27] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.-L. Barabási, The large-scale organization of metabolic networks, Nature **407** (2000) 651-654.

[28] P. L. Krapivsky, S. Redner, F. Leyvraz, Connectivity of growing random networks, Phys. Rev. Lett. **85** (2000) 4629-4632.

[29] F. Liljeros, C.R. Edling, L.A.N. Amaral, H.E. Stanley, and Y.Åberg, The web of human sexual contacts, Nature **411** (2001) 907-908.

[30] M.E.J. Newman, The structure of scientific collaboration networks, Proc. Natl. Acad. Sci. U.S.A. **98** (2001) 404-409.

[31] M.E.J. Newman, The structure and function of complex networks, SIAM Review **45** (2003) 167-256.

[32] D.J. Watts and H. Strogatz, Collective dynamics of 'small-world' networks, Nature **393** (1998) 440-442.

[33] Z.Z. Zhang, F. Comellas, G. Fertin, and L.L. Rong, High Dimensional Apollonian Networks, J. Phys. A, **39** (2006) 1811-1818.

[34] Z.Z. Zhang, L.L. Rong and F. Comellas, Evolving small-world networks with geographical attachment preference, J. Phys. A **39** (2006) 3253-3261.

[35] Z.Z. Zhang, L.L Rong and F. Comellas, High dimensional random Apollonian networks. Physica A **364**, (2006) 610-618.

[36] Z.Z. Zhang, L.L Rong and C.H. Guo, A deterministic small-world network created by edge iterations, Physica A **363** (2006) 567-572.

[37] Z.Z. Zhang, L.L Rong and S.G. Zhou, Evolving Apollonian networks with small-world scale-free topologies, Phys. Rev. E **74**, (2006) 046105.

[38] Z.Z. Zhang and S.G. Zhou, Correlations in Apollonian networks, Preprint cond-mat/ 0609270.

[39] Z.Z. Zhang, L.L. Rong, Shuigeng Zhou, and Zhiyong Wang, (unpublished)

[40] T. Zhou, G. Yan, and B.H. Wang, Maximal planar networks with large clustering coefficient and power-law degree distribution, Phys. Rev. E **71**, (2005) 046141.